Borsuk-Ulam theorem in Combinatorics

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Abstract

Here is the outline of the talk I will be giving to the Graduate Student Seminar (at Yale). The talk will be about the Borsuk Ulam theorem and its applications to discrete mathematics problems. Most of the proofs written below will be sketches, and will not go into painful details.

1 The theorem

Theorem 1. *For every $n \geq 0$, we have for every continuous map $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.*

2 Direct Applications of Borsuk-Ulam

2.1 The Ham Sandwich Theorem

Intuitively speaking, the HST says that if we make a sandwich with the following three ingredients: Cheese, ham and bread, and we stack them however we want, and we place as many slices as we want, that then there exists a cut such that it splits the sandwich such that both parts have the same amount of ham, bread and cheese (all three items are split evenly simultaneously!). Formally speaking, here is the theorem:

Theorem 2. *Let $\mu_1, ..., \mu_d$ be finite Borel measures on $\mathbb{R}^d$ such that every hyperplane has measure 0 for each of the μ_i. Then there exists a hyperplane h such that*

$$\mu_i(h^+) = \frac{1}{2}\mu_i(\mathbb{R}^d)$$

for $i = 1, ..., d$. Here h^+ denotes one of the half spaces defined by h.

Proof. The idea is as follows, for each point in S^d assign a hyper plane in $\mathbb{R}^d$ as follows:

$$u \mapsto h^+(u)$$

$$(u_0, ..., u_d) \mapsto \{(x_1, ..., x_d) \in \mathbb{R}^d : u_1x_1 + ... + u_dx_d \leq u_0\}$$

before we go on, note that $h^+(-u) = \{(x_1, ..., x_d) \in \mathbb{R}^d : -u_1x_1 - ... - u_dx_d \leq -u_0\} = \{(x_1, ..., x_d) \in \mathbb{R}^d : u_1x_1 + ... + u_dx_d \geq u_0\}$, so antipodal points correspond to opposite half spaces. Now define a function from S^d to $\mathbb{R}^d$ as follows:

$$f : u \mapsto (\mu_1(h^+(u)), ..., \mu_d(h^+(u)))$$

Note that by Borsuk Ulam, there exists a point u such that $f(u) = f(-u)$, so indeed we have $\mu_i(h^+(u)) = \mu_i(h^+(-u))$ which by previous remarks we have $h^+(-u) = \mathbb{R}^d \backslash h^+(u)$, and since hyperplanes have measure zero: $\mu_i(h^+(u)) = \frac{1}{2}\mu_i(\mathbb{R}^d)$. □

Now we give the discrete version of the HST,

Theorem 3. *Let $A_1, ..., A_d$ be finite point sets in $\mathbb{R}^d$. Then there exists a hyperplane h that simultaneously bisects $A_1, ..., A_d$.*

Careful, since we don't know whether the points are in general position or not, the word bisect means that each of the open half spaces contains at most $|\frac{1}{2}A_i|$ points of A_i. If however, the points are in general position and the sets A_i are disjoint, we can say that each of the open half spaces contains half of the points (with one point in h if $|A_i|$ is odd).

2.2 Multicolored Partitions

Consider the following game: You have n red points in the black board and n blue points in the board. Can you always pair them up such that the straight lines joining any two of them are non-intersecting? What about for higher dimensional cases? The answer is yes in both cases:

Theorem 4. *Let $A_1, ..., A_d$ be disjoint sets of cardinality n living in $\mathbb{R}^d$ (in general position). Let us think of the sets A_i as having different colors. Then there exists a partition of the set $A_1 \cup ... \cup A_d$ into n rainbow d-tuples whose convex hulls are disjoint.*

Proof. If n is odd, by the discrete HST in general position, we have that there exists a hyperplane that contains exactly one point of each A_i and then we can apply induction to each half. For the even case, it is the same, but our hyperplane won't contain any points. The case for $n = 2$ is easy: Consider a matching such that the sum of the lengths of the lines is minimal. If there are two that intersect, contradict minimality.

□

2.3 Necklace Theorem

Imagine two thieves that steal a (open) necklace. The necklace contains a bunch of beautiful stones (diamonds, saphires, etc), and say that there is an even number of each kind, and a total of d different kinds of stones. Then they want to split the loot, so they have to make cuts to the necklace, the question is: What is the minimum number of cuts that are required for them to split the loot?

Theorem 5. *At most d cuts are necessary*

Proof. Place the necklace into the moment curve $\gamma(t) = (t, t^2, ..., t^d)$ (parametric equation). If the necklace has n stones, we define:

$$A_i = \{\gamma(k) : \text{ the kth stone is of the ith kind } k = 1, 2, .., n\}$$

Then apply the HST (discrete), and note that since the points are in a moment curve, they are already in general position. Thus, there exists a hyperplane that cuts divides evenly each of the A_i. This hyperplane can only intersect the moment curve in at most d places (fact about moment curves), so we get the desired result. □

2.4 Chromatic number of the Kneser Graph

Let us define the kneser graph, $KG_{n,k}$. The set of vertices are the k-subsets of $[n]$ and they are joined by the following rule: Two vertices are adjacent if and only if the corresponding sets are disjoint. Note a couple of interesting examples: If $2k > n$ then we have that any two distinct k-subsets will have to intersect, so we would have a graph with no edges (boring), so consider cases when $2k \leq n$. Note that if we have equality, i.e., $2k = n$, then we have a matching (since every set will only be adjacent to its complement), again not very interesting.

Define the chromatic number to be the minimum number of colors required to color the vertex set of a graph such that adjacent vertices get different colors.

The conjecture made by Kneser in 1955, and proved by Lováz is the following:

Theorem 6. $\chi(KG_{n,k}) = n - 2k + 2$ *for* $n \geq 2k - 1$.

Before going into the proof of the theorem, let us define a new term and make a remark.

Define $\chi_f(G)$ (the fractional chromatic number of G) to be the infimum of the set $\{\frac{a}{b} \mid$ the vertex set $V(G)$ can be covered by a independent sets, and each vertex is covered at least b times$\}$

It is easy to see that $\chi_f(G) \leq \chi(G)$, but it is not easy to see whether this two values are going to be close to each other. Kneser graphs provide a family of examples such that χ_f is much smaller than χ (not many examples are known).

Lemma 1. $\chi_f(KG_{n,k}) = n/k$ *for* $n \geq 2k$.

Proof. The reason why I want to prove this is because it uses one of my favorites theorems in combinatorics: Erdos-Ko-Rado theorem. It says that if we have $\mathcal{F} \subset 2^{[n]}$ a family of k-sets, with the property that any two sets $F, F \in \mathcal{F}$ intersect, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ (in fact this is tight by considering all the k-sets that contain a fixed element).

First we prove that $\chi_f(KG) \leq n/k$: Consider the sets A_i to be the k-subsets of $[n]$ that contain i. Then note that A_i is an independent set since any two members of A_i contain i (and hence not an empty intersection), so there is no edges in between them. Secondly, then we have $A_1, ..., A_n$, and note that each vertex is covered exactly k times (for instance the vertex $\{1, 2., ..., k\}$ is covered by $A_1, ..., A_k$). Hence, $\chi_f \leq n/k$.

For the other direction we will need the Erdos-Ko-Rado theorem: Say we can cover KG by a independent sets $A_1, .., A_a$ and each vertex is covered at least b times. Then look at $\bigcup A_i$ as a multiset. Let us bound above and below $|\cup A_i|$. Since each vertex gets covered at least b times and there are $\binom{n}{k}$ vertices we have that:

$$b\binom{n}{k} \leq |\cup A_i|$$

to bound it above note that each $|A_i|$ is an independent set, so each A_i satisfy the conditions of the EKR theorem, hence:

$$|\cup A_i| \leq a\binom{n-1}{k-1}$$

putting these two together:

$$\frac{n}{k} = \frac{\binom{n}{k}}{\binom{n-1}{k-1}} \leq \frac{a}{b}$$

and hence, $\chi_f(KG) \geq n/k$. □

Proof. **Theorem 6:**In order to do this proof we need a consequence (it is actually an equivalent statement, but that doesn't concern us) of the Borsuk Ulam theorem.

Lemma 2. ***Lyusternik-Shnirel'man:****For any cover $F_1, .., F_{n+1}$ of S^n by $n+1$ closed sets, tehre is at least one set containing a pair of antipodal points. The statement also holds if the sets F_i are all open. The statement also holds if the F_i are either open or closed.*

Proof. Closed case: Define $f : S^n \to \mathbb{R}^n$ by $f(x) = (d(x, F_1), ..., d(x, F_n))$, then there exists a point with $f(x) = f(-x) = y$. If any of the coordinates of $y = 0$, then we are done. If all the coordinates are non-zero, then they lie in F_{n+1}.

Open case: Consider an open cover $U_1, .., U_{n+1}$, then for each U_i there is a closed $F_i \subset U_i$ such that F_i is a closed set and they are a covering of the sphere. For the mixed case do the same to the open sets of the collection. □

With this in mind we go ahead and prove Kneser's conjecture: First we show that it is impossible to color the graph with at most $d = n - 2k + 1$ colors. Consider a set X of n points in S^d such that any hyperplane going through the origin in $\mathbb{R}^{d+1}$ does not contain more than d points. Then let us think of the set X as $[n]$. Assume that there is a proper coloring, so for every k-tuple of X we assign a color. Construct the sets $A_1, ..., A_d$ by the following rule, the point $x \in S^d$ is in A_i if there is a k-tuple of points in X in the open hemisphere, $H(x)$, that gets assigned the color i. Then $A_1, ..., A_d$ are d open sets, and let A_{d+1} be the complement of them (a closed set). Then we have a covering, and by previous remarks there is an index i such that $x, -x \in A_i$.

If $i \leq d$, then it means that there is a k-tuple of color i, and a disjoint k-tuple of color i as well. This is a contradiction since we have a proper coloring.

If $i = d + 1$, then we have that $H(x)$ and $H(-x)$ do not contain a k-tuple (otherwise x and $-x$ would be in some A_j with $j < d + 1$). Then the set $S^d \backslash (H(x) \cup H(-x))$ contains at least $n - (k-1) - (k-1) = n - 2k + 2 = d + 1$ points, but this contradicts the choice of X since this would imply that there is a hyperplane through the origin containing more than d points. Hence, we cannot color KG using at most $n - 2k + 1$ colors.

Now we have to show that $n - 2k + 2$ colors suffice: Color the vertices F of KG by the following rule:

$$\chi(F) = \min\{\min(F), n - 2k + 2\}$$

If two sets F and F' get the same color, and it is less than $n - 2k + 2$, then they intersect, so they are disjoint. If they get the same color and the color is $n - 2k + 2$, then both F and F' are contained in $\{n - 2k + 2, ..., n\}$ a set with $2k - 1$ and so the sets must intersect, hence they are not adjacent.

□

Now note that if we let $n = 3k$, by our first part we obtain that $\chi_f(KG) = 3$ whereas $\chi(KG) = k + 2$, so as promised we found a family of graphs for which we can make the gap between fractional chromatic number and chromatic number as big as possible.