NOTES FOR THE ANALYSIS QUAL

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ABSTRACT. This are some notes I am typing for the analysis qual. I have received some help from other current graduate students. I will be focusing on professor Margulis old quals since he will be the one giving it out this coming spring.

1. THEOREMS TO KNOW (STATEMENT AND PROOF)

1 Monotone Convergence Theorem
2 Fatou's Lemma
3 Dominated Convergence Theorem
4 Open Mapping Theorem
5 Riemann Mapping Theorem
6 Banach algebra, elements have non-empty spectrum
7 Baire Category Theorem
8 Spectral Theorem for self-adjoint compact operators
9 Holder's inequality
10 Minkowsky inequality
11 Minkowsky inequality for integrals
12 Extreme Points for L^1 and L^∞
13 $\|T\| = \|T^*\|$

1.1. **Monotone Convergence Theorem.**

Theorem 1. *Let X be a measurable space. Let $\{f_n\}$ be a monotonic sequence (i.e., $f_n(x) \leq f_{n+1}(x)$ for all $x \in X$) of non-negative measurable functions which converge pointwise to a function f. Then we have that f is measurable and moreover that $\lim \int f_n = \int f$.*

Proof. It is easy to see that f is measurable. To see that the limit commutes with the integral consider the following: As $f_n \leq f_{n+1}$ then we have that $f_n \leq f$, by taking integrals we get that $\int f_n$, since this holds for every n, we have that

$$\lim \int f_n \leq \int f$$

For the other direction, let α be any positive number strictly less than 1, and let ϕ be a simple function such that $\phi \leq f$. Then let $E_n = \{x \mid f_n(x) \geq \alpha\phi(x)\}$. Since $\{f_n\}$ is a monotonic sequence note that $E_1 \subset E_2 \subset E_3 \ldots$. Also note that since $f_n \to f$ we have that $\cup E_i = X$. Hence,

$$\int f_n \geq \int_{E_n} f_n \geq \int_{E_n} \alpha\phi(x) = \alpha \int_{E_n} \phi(x)$$

as $n \to \infty$ we have

$$\lim \int f_n \geq \alpha \int_{E_n} \phi(x) = \alpha \int \phi(x)$$

Since this holds for every $\alpha < 1$, it also holds for $\alpha = 1$, and taking supremum over all $\phi(x)$ yields:

$$\lim \int f_n \geq \int f$$

just as desired. $\square$

1.2. **Fatou's Lemma.**

Theorem 2. *Let $\{f_n\}$ be a sequence of non-negative measurable functions, then*

$$\int (\liminf f_n) \leq \liminf \int f_n$$

Proof. The idea is to use the Monotone Convergence theorem. Hence, we have to construct a monotonic sequence of functions. The natural way is to define $g_k = \inf_{i \geq k} f_i$. This yields $g_k \leq g_{k+1}$ and moreover we have that $\lim_{k \to \infty} g_k = \liminf f_n$. Hence,

$$\lim \int g_k = \int \lim g_k = \int \liminf f_n$$

where the first equality above holds by MCT. Also, note that since $g_k \leq f_k$, so we have that $\int g_k \leq \int f_k$. Take $\liminf$ at both sides to obtain:

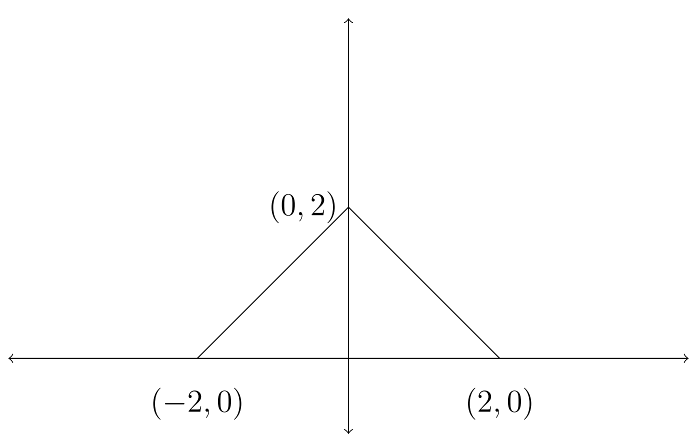
$$\liminf \int g_k \leq \liminf \int f_k$$

but $\{\int g_k\}$ has a limit, so $\lim inf$ agrees with $\lim$, and by the line above we get:

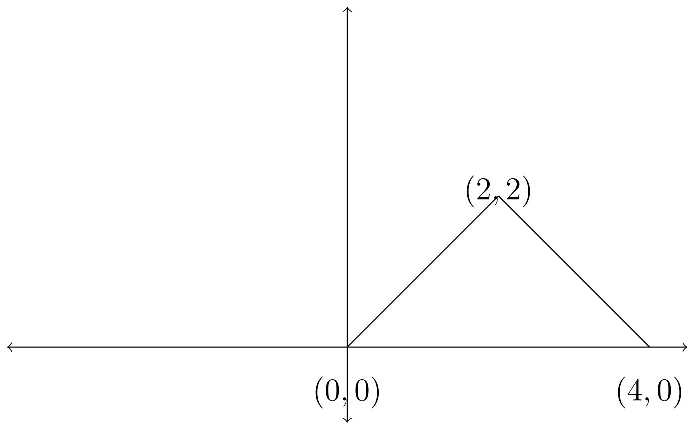
$$\int \liminf f_n \leq \liminf \int f_n$$

$\square$

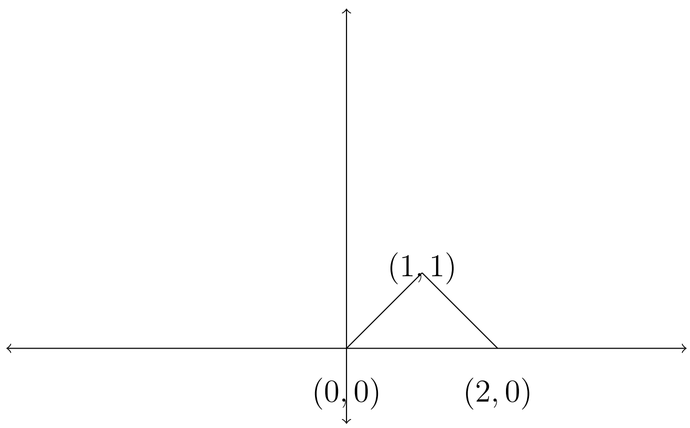
It is important to note that strict inequality can hold. Consider f_{2n+1} to be:



and f_{2n} to be



then $\int f_n = 4$, so $\liminf \int f_n = 4$. Note however that $\liminf f_n$ is:



and we get that $\int \liminf f_n = 1$. Hence strict inequality hapens.

1.3. **Dominated Convergence theorem.** Suppose $\{f_n\}$ is a sequence of measurable functions that converge pointwise to f. If there exists a function $g \in L^1$ such that

$$|f_n(x)| \leq g(x) \qquad (n = 1, 2, ...; x \in X)$$

then $f \in L^1$ and

$$\lim \int |f_n - f| = 0$$

$$\lim \int f_n = \int f$$

Before we begin the proof let me state a couple of things. The convergence of the $f_n \to f$ does not have to be in X. It can happen a.e. and the result will be the same.

Proof. First of all note that since $|f_n| \leq g$, then we have that $|f| \leq g$, so in particular we will have that $f \in L^1$. Now note that $|f - f_n| \leq 2g$, so we have:

$$2g = 2g - \liminf |f - f_n|$$

since $\liminf |f - f_n| = 0$. Taking integrals at both sides:

$$\int 2g = \int (2g - \liminf |f - f_n|) = \int \liminf(2g - |f - f_n|) \leq \liminf \int 2g - |f - f_n|$$

where the above inequality is given by Fatou's. Then,

$$= \int 2g + \liminf(-\int |f - f_n|) = \int 2g - \limsup \int |f - f_n|$$

Thus,

$$\limsup \int |f - f_n| \leq 0$$

a sequence of non-negative numbers whose limit supremum is non-positive must converge to 0. That is, $\lim \int |f - f_n| = 0$, just as desired. From here we get that $|\int f - f_n| \leq \int |f - f_n| \to 0$, so $\int f_n \to \int f$, which is the second conclusion. □

1.4. **Open Mapping theorem.**

Theorem 3. *Let $T : X \to Y$ be a surjective linear map between Banach spaces. Then T is open.*

Proof. First of all note that it suffices to show that $T(B_1)$ has an open ball around 0 (why?). Denote by B_ϵ the open ball of radius ϵ. Then we have that $\cup_{n=1}^{\infty} T(B_n) = Y$ since T is surjective. By Baire's category theorem, we have that there is an n such that $T(B_n)$ is not nowhere dense. By contraction, we can assume that $T(B_1)$ is not nowhere dense. Hence, there exists an open set W such that $W \subset \overline{T(B_1)}$. Let $y_0 \in W$ and $r > 0$ be such that $B(y_0, 4r) \subset W$ (ball centered at y_0 or radius $4r$). Then let $y_1 = T(x)$ be such that $|y_1 - y_0| \leq 2r$. Then for any $y \in Y$ with $|y| < 2r$ we have:

$$y = y_1 + y - y_1 = T(x_1) + (y - y_1)$$

note $y - y_1 \in B(y_1, 2r)$ since $|y| < 2r$, so we have that

$$y = T(x_1) + (y - y_1) \subset T(B_1) + B(y_1, 2r) \subset T(B_1) + B(y_0, 4r) \subset T(B_1) + \overline{T(B_1)} = \overline{T(B_2)}$$

dividing by 2, we obtain that if $|y| < r$ then $y \in \overline{T(B_1)}$. In general, $|y| < r/2^n$ will be in $\overline{T(B_{1/2^n})}$.

Assume that $|y| < r/2$, then $y \in \overline{T(B_{1/2})}$, so let $x_1 \in B_{1/2}$ be such that $|y - Tx_1| < r/4$. Then we have that $y - Tx_1 \in \overline{T(B_{1/4})}$, so let $x_2 \in B_{1/4}$ be such that $|(y - Tx_1) - Tx_2| < r/8$. Then we have that $y - Tx_1 - Tx_2 \in \overline{T(B_{1/8})}$. Recursively, obtain $x_n \in B_{1/2^n}$ to be such that $|y - \sum_{i=1}^{n} T(x_i)| < r/2^{n+1}$. Note that since $|\sum_{i=1}^{\infty} x_i| \leq \sum_{i=1}^{\infty} |x_i| < 1$, we have that $\sum x_i$ converges to a point x (as X is Banach), so in particular we have that $y = Tx$, and since $x \in B_1$, we obtain that $y \in T(B_1)$. That is, we have shown that $B_{r/2} \subset T(B_1)$. □

1.5. **Riemann Mapping Theorem.**

Theorem 4. *Let Ω be a simply connected domain of $\mathbb{C}$ which is proper. Then there exists a biholomorphic map from Ω to the unit disk.*

Proof. We will split the proof into three steps. Some of the steps do not go into horrifying detail since a sketch is what is sufficient and expected.

1 First we will prove that we can map Ω into a subset of the unit disk containing 0. Hence, for the next two steps we will assume that $\Omega \subset \mathbb{D}$.
2 We will create a family of injective functions from Ω to $\mathbb{D}$ with the condition that $f(0) = 0$. We will look at the $\sup\{|f'(0)|\}$, and we will show that there is a function in our family that attains such a maximum using Montel's theorem.

3 We will show that the function that we obtain in step 2 is actually surjective, and hence the desired map we were looking for. We will do so by supposing that it is not surjective, and then construction a function with a higher derivative at zero, which would contradict the maximality condition that it ought to have.

Step 1: First of all since our Ω is proper, there is a α such that $z-\alpha$ does not vanish in Ω, since it is also simply connected, we have that we can define a branch of the logarithm here:

$$f(z) = \log(z-\alpha)$$

by exponentiating both sides we see that f is an injective function. Secondly, pick a point $w \in \Omega$, we claim that there is a circle centered at $f(w)+2\pi i$ that does not meet $f(\Omega)$. To see this, assume that there is a sequence of points z_n such that $f(z_n) \to f(w)+2\pi i$. Then we have by taking $\exp$ at both sides that $z_n \to w$ so by taking f again we get: $f(z_n) \to f(w)$, a contradiction. Hence, there is a δ such that $d(f(w)+2\pi i, f(\Omega)) > \delta$. Define:

$$F(z) = \frac{1}{f(z) - (f(w)+2\pi i)}$$

hence, we have that $|F(z)| < \delta^{-1}$, so we have that $F(\Omega)$ is an injective bounded function. By translationg and scaling, we can assume that F maps Ω into $\mathbb{D}$ and $F(0)=0$.

Step 2: Assume that Ω contains 0 and is contained in $\mathbb{D}$ by virtue of the previous step. Define:

$$\mathcal{F} = \{f : \Omega \to \mathbb{D} \mid f \text{ is holomorphic and injective and } f(0)=0\}$$

note that $\mathcal{F}$ is not empty since it contains the identity. Now consider $c = \sup\{|f'(0)|\}$. Let $\{f_n\}$ be a sequence of functions such that $|f_n'(0)| \to c$. Using Cauchy's inequality we get that $\mathcal{F}$ is uniformly bounded, so by Montel's theorem, we get that there is a subsequence of $\{f_n\}$ that converges uniformly on every compact set. By relabeling, assume that $\{f_n\}$ does converge uniformly to the function f. Note that $|f'(0)'| = c$, and to see that $f \in \mathcal{F}$ we have to check a couple of things. Since f is the uniform limit of injective functions, we have that f is either constant or injective. It cannot be injective since $c \geq 1$ as the identity in in $\mathcal{F}$. Thus, f is injective. Also, we have that $|f(z)| \leq 1$, but by maximum modulus, we have that $|f(z)| < 1$. The last condition is clear, $f(0)=0$.

Step 3: We claim that the function f from step 2 is surjective. Assume it is not surjective, then there is an $\alpha \in \mathbb{D}$ such that α is not in the image of f. Then let φ_β be the FLT that interchanges 0 and β and let g be the square root function (which we can define in a simply connected domain that does not contain 0). Consider the function:

$$F = \varphi_{g(\alpha)}^{-1} \circ g \circ \varphi_\alpha \circ f$$

It is easy to see that F is injective that that F takes 0 to 0. Solving for f we get:

$$f = \Phi \circ F$$

Note that Φ is not injective, so by Schwarz lemma, we have that $|\Phi'(0)| < 1$. Hence, $|F'(0)| > |f'(0)|$, a contradiction with the choice of f. Hence, f is indeed surjective. $\square$

1.6. **Banach Algebra, elements have a non-empty spectrum.** The question usually asks to define a Banach algebra:

Definition 5. *We say A is a Banach algebra if it is a Banach space, where we define a multiplication in A that satisfies the following properties:*

$$\|xy\| \leq \|x\|\|y\|$$

$$x(a+b) = xa + xb \qquad (a+b)x = ax + bx \qquad x(\alpha y) = \alpha(xy) = (\alpha x)y \qquad (xa)b = x(ab)$$

where $x, a, b \in A$ and $\alpha \in \mathbb{F}$.

We usually assume that $\mathbb{F} = \mathbb{C}$ and that A contains a unit element e such that $xe = x = ex$ for all $x \in A$. Define $\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda e$ is not invertible$\}$

Theorem 6. *Let A be a Banach algebra (over the complex numbers) with unit. Then for any $x \in A$, $\sigma(x)$ is not empty.*

Proof. Let $x \in A$, and assume for sake of a contradiction that $\sigma(x) = \emptyset$, then we have that for $\lambda_0 \in \mathbb{C}$, $(x - \lambda_0 e)^{-1} \neq 0$, so define $\phi((x - \lambda_0 e)^{-1}) = \|(x - \lambda_0 e)^{-1}\|$. Then ϕ extends to a bounded linear functional in all of A by Hanh-Banach. Define F to be the following map:

$$F(\lambda) = \phi((x - \lambda e)^{-1})$$

where the above map is well defined for all $\lambda \in \mathbb{C}$ since we are assume that $\sigma(x)$ is empty. Now note that

$$F(\lambda) = \phi((x - \lambda e)^{-1}) = (1/\lambda)\phi((x/\lambda - e)^{-1})$$

as $\lambda \to \infty$, we have that $x/\lambda - e \to e$, so $\phi((x/\lambda - e)^{-1}) \to \phi(-e)$, and we see that $F(\lambda) \to 0$. If we manage to show that F is holomorphic, then we would have that F is entire and by Liouville's theorem it is constant, but it is constant that tends to 0, so it is the zero function, but that is a contradiction since ϕ is not the zero functional.
Hence, it remains to prove that F is indeed holomorphic:

$$\lim_{h \to 0} \frac{|F(\lambda + he) - F(\lambda)|}{|h|}$$

□

1.7. **Baire Category Theorem.**

1.8. **Spectral Theorem for self-adjoint compact operators.**

Theorem 7. *Let H be a Hilbert space (non-empty), and say T is a compact self-adjoint linear operator. Then there exists an orthonormal basis consisting of eigenvectors. Moreover, we have that for every $\epsilon > 0$, we only have finitely many (counting multiplicity) eigenvalues outside of the ball of radius ϵ*

Proof. We first show that H must contain an eigenvector. First of all note that $\langle Tx, x\rangle \in \mathbb{R}$ since T is self-adjoint. Then use the result that says that for T self-adjoint we have:

$$\|T\| = \sup\{|\langle Tx, x\rangle| : \|x\| \leq 1\}$$

Then, let x_n be a sequence in B_1 (open ball of radius 1), such that $|\langle Tx_n, x_n\rangle| \to \|T\|$, and let λ be such that $\langle Tx_n, x_n\rangle \to \lambda$, by previous remarks we have that $\lambda \in \mathbb{R}$, and we have that $|\lambda| = \|T\|$. Then we have that $\{Tx_n\}$ is a sequence in $T(B_1)$, so by compactness of T, we have that there exists a subsequence that converges. Up to relabeling, we can assume that $\{Tx_n\}$ converges, to say y. Note that

$$\|Tx_n-\lambda x_n\|^2 = \langle Tx_n-\lambda x_n, Tx_n-\lambda x_n\rangle = \|Tx_n\|^2+\|\lambda Tx_n\|^2-2\lambda\langle Tx_n, x_n\rangle \leq 2\lambda^2-2\lambda\langle Tx_n, x_n\rangle$$

which tends to 0. Hence, we have that $Ty = \lambda y$ because:

$$Ty = T(\lim Tx_n) = T(\lim \lambda x_n) = \lambda \lim T(x_n) = \lambda y$$

(second equality is due to the previous remark). Hence, we have that y is an eigenvalue.

Using Zorn's lemma, choose a maximal set of orthonormal eigenvectors, call it E. Let W be the closure of the span of E. We claim that W is equal to H. For sake of contradiction that W is properly contained in H. Then $W^\perp$ is not empty. It is easy to see that W is T-invariant, which makes $W^\perp$ T-invariant as well. We can restric T to $W^\perp$, and we still have a compact, self-adjoint operator. Then applying the same argument as above, we have that we can find an eigenvalue. A contradiction with the maximality given by Zorn's. Hence, $W = H$, just as desired.
For the last remark let $\epsilon > 0$, and for sake of contradiction say that there are infinitely many eigenvalues with multiplicity outside the ball of radius epsilon. Then for any two eigenvectors $v_1 \neq v_2$ with corresponding eigenvalues λ_1, λ_2 we have that

$$\|Tv_1 - Tv_2\|^2 = \|Tv_1\|^2 + \|Tv_2\|^2 = \lambda_1^2 + \lambda_2^2 > 2\epsilon^2$$

so since their distance is bounded below an infinite sequence will never have a convergent subsequence, that is a contradiction since T is supposed to be compact. □

1.9. **Holder's inequality.**

Theorem 8. *Suppose $1 < p < \infty$ and p, q Holder conjugates. If f and g are measurable functions on X, then*

$$\|fg\|_1 \leq \|f\|_p\|g\|_q$$

Proof. We will use Young's inequality which states that if $u, v > 0$ then we have that $uv \leq u^p/p + v^q/q$ where p, q are Holder conjugates. Assuming this the rest of the problem is very easy: First of all note that if $\|f\|_p = 0$ then $f = 0$ a.e., and so $fg = 0$ a.e, so we have the equality $0 = 0$, also if $\|f\|_p = \infty$ then it holds trivially (this is an abuse of notation, when people write $\|f\|_p$ then assume that $f \in L^p$ and thus it is a finite quantity). Similarly for g. Then we can define:

$$u = \frac{|f|}{\|f\|_p} \qquad v = \frac{|g|}{\|g\|_q}$$

Applying Young's inequality:

$$\frac{|fg|}{\|f\|_p\|g\|_q} = uv \leq \frac{u^p}{p} + \frac{v^q}{q} = \frac{|f|^p}{p\|f\|_p^p} + \frac{|g|^q}{q\|g\|_q^q}$$

integrating both sides we get:

$$\frac{\int |fg|}{\|f\|_p\|g\|_q} \leq 1$$

as the desired result follows. □

1.10. **Minkowski inequality.**

Theorem 9. *Let $f, g \in L^p$. Then we have that $f+g \in L^p$ and moreover,*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Proof. First we have to show that $f+g \in L^p$. To do so, consider the following:

$$|f+g|^p = |(2f)/2 + (2g)/2|^p \leq |2f|^p/2 + |2g|^p/2$$

where the inequality comes from the convexity of x^p. Thus,

$$|f+g|^p \leq 2^{p-1}(|f|^p + |g|^p)$$

integrating at both sides shows us that $f+g \in L^p$. Now,

$$|f+g|^p = |f+g||f+g|^{p-1} \leq (|f|+|g|)|f+g|^{p-1}$$

integrating both sides:

$$\|f+g\|_p^p \leq \int (|f|+|g|)|f+g|^{p-1} = \int |f||f+g|^{p-1} + \int |g||f+g|^{p-1}$$

$$\leq (\|f\|_p + \|g\|_p)\left(\int (|f+g|^{p-1})^q\right)^{1/q} = (\|f\|_p + \|g\|_p)\left(\int (|f+g|^p\right)^{(p-1)/p}$$

$$= (\|f\|_p + \|g\|_p)\|f+g\|_p^{p-1}$$

and the result follows. ∎

1.11. **Minkowsky inequality for integrals.** Let (X, Σ_1, μ) and (Y, Σ_2, ν) be σ-finite measure spaces. Let $f : X \times Y \to [0, \infty]$ be a measurable function, and let $1 \leq p < \infty$. Then:

$$\left\{\int_X \left(\int_Y f(x,y)d\nu(y)\right)^p d\mu(x)\right\}^{1/p} \leq \int_Y \left(\int_X f(x,y)^p d\mu(x)\right)^{1/p} d\nu(y)$$

Proof. Note that if $p=1$, then the theorem is just Tonelli's, so assume $p>1$. First of all we will consider the case when the function defined by $F(x) = \int_Y f(x,y)d\nu(y)$ is in L^p. Using this result, we can always break the general case into this one.

Let $g \in L^q$ be any function with $\|g\|_q \leq 1$. Then,

$$\int_X F(x)|g(x)|d\mu(x) = \int_X \int_Y f(x,y)d\nu(y)|g(x)|d\mu(x) = \int_Y \int_X f(x,y)|g(x)|d\mu(x)d\nu(y)$$

$$\leq \int_Y \left(\int_X f(x,y)^p d\mu(x)\right)^{1/p} \left(\int_X |g(x)|^q d\mu(x)\right)^{1/q} d\nu(y) = \|g\|_q \int_Y \left(\int_X f(x,y)^p d\mu(x)\right)^{1/p} d\nu(y)$$

$$\leq \left(\int_X f(x,y)^p d\mu(x)\right)^{1/p} d\nu(y)$$

where on the second equality we used Tonelli's and the inequality comes from Holder's. Taking supremum over $g \in L^q$ with norm bounded by one we get:

$$\|F(x)\|_p \leq \left(\int_X f(x,y)^p d\mu(x)\right)^{1/p} d\nu(y)$$

which is what we wanted to prove.

To prove it for the general case.Consider $X_1 \subset X_2 \subset X_3 \subset ...$ and $Y_1 \subset Y_2 \subset Y_3 \subset ...$ such that $\cup X_i = X$, $\cup Y_i = Y$ and X_n and Y_n of finite measure. Then define $f_n(x, y) = \min\{n, f(x, y)\}$ in the space $X_n \times Y_n$. Hence, we have that $f_n \to f$ pointwise and monotonic. Apply above to each f_n and the result follows. $\square$

1.12. **Extreme points of $L^1[0, 1]$ and L^∞[0,1].** Below denote by B_1 the closed unit ball. Denote $Ext(C)$ the set of extreme points of the set C.

Theorem 10. *In L^∞ we have that $Ext(B_1) = \{f : |f| = 1$ a.e. $\}$*

Proof. Assume that f is in B_1 and that $|f| \neq 1$ a.e., then there must exist a set with positive measure such that $|f| < 1$. That is, $E = \{x : |f(x)| < 1\}$ has positive measure. Define $E_n = \{x : |f(x)| \leq 1 - 1/n\}$, then we have $E = \cup E_n$, and since E has positive measure, there exists a set E_n with positive measure. Then define:

$$h(x) = \begin{cases} f(x) + \frac{1}{n}, & E_n \\ f(x), & E_n^c \end{cases} \qquad g(x) = \begin{cases} f(x) - \frac{1}{n}, & E_n \\ f(x), & E_n^c \end{cases}$$

note that h, g are both in B_1 and moreover $f = \frac{1}{2}h + \frac{1}{2}g$, and $h \neq f$ since they differ on a set with positive measure. Hence, f is not an extreme point. Ergo, we have shown $Ext(B_1) \subset \{f : |f| = 1$ a.e.$\}$.

For the other inclusion, assume that f is such that $|f| = 1$ a.e., and write f as a convex combination of two function in B_1:

$$f(x) = \lambda h(x) + (1 - \lambda)g(x)$$

taking norms at both sides:

$$|f(x)| = 1 = |\lambda h(x) + (1 - \lambda)g(x)| \leq \lambda|h(x)| + (1 - \lambda)|g(x)| \leq 1$$

hence, the inequalities above are actually equalities. For that to hold we must have that $|h| = 1 = |g|$ a.e., so we see that h and g take values on the unit circle in $\mathbb{C}$, this is a stricly convex space, so if there is a set with positive measure, E, such that $h \neq g$ for $x \in E$, then $\lambda h(x) + (1 - \lambda)g(x)$ would be a point in the chord matching two distinct points in S_1, so it would have norm strictly less than 1, a contradiction because $|f|$ cannot be less than 1 in a set of positive measure. Hence, $g = h$ a.e., and it follows that $f = g = h$ a.e., and we conclude that f is an extreme point. The desired result has been proven. $\square$

Theorem 11. *B_1 has no extreme points in $L^1[0, 1]$.*

Proof. We first show that if B_1 does have extreme points, then they ought to be of norm 1: Let $f \in B_1$ $(0 \neq f)$ and say that $\|f\| < 1$ (I am using $\|.\|$ instead of $\|.\|_1$ because this part of the proof does not require the fact that we are in L^1), then let $\epsilon > 0$ be such that $\|f\| + \epsilon \leq 1$ and $\|f\| > \epsilon$. Then we have:

$$f = \frac{1}{2}\left(\frac{\|f\| + \epsilon}{\|f\|} \cdot f\right) + \frac{1}{2}\left(\frac{\|f\| - \epsilon}{\|f\|} \cdot f\right)$$

the conditions on ϵ guarantee that $\frac{\|f\|+\epsilon}{\|f\|} \cdot f$ and $\frac{\|f\|-\epsilon}{\|f\|} \cdot f$ will be in B_1. Hence, f is not an extreme point.

Now we will need the fact that we are in L^1. Assume that f is in B_1 and for sake of contradiction that it is an extreme point. Then by the above argument, we have that $\|f\|_1 = 1$, that is:

$$1 = \int_0^1 |f|$$

Define

$$F(t) = \int_0^t |f|$$

then we have that F is a continous function, so we have that there exists a $t \in (0,1)$ such $F(t) = 1/2$. Then,

$$h(x) = \begin{cases} 2f(x), & [0,t] \\ 0 & (t,1] \end{cases} \qquad g(x) = \begin{cases} 0, & [0,t] \\ 2f(x), & (t,1] \end{cases}$$

we have that $\|h\|_1 = \|g\|_1 = 1$, $f \neq h$ and $f = (1/2)h + (1/2)g$, a contradiction. Hence, B_1 has no extreme points. $\square$

1.13. $\|T\| = \|T^*\|$.

Theorem 12. *Let X be a Banach space, and let T be a bounded linear operator, then $\|T\| = \|T^*\|$*

Proof. The result is really easy if we assume that for an element x we have:

$$\|x\| = \sup\{|f(x)| : f \in B_1^*\}$$

where B_1 is the closed unit ball in X and B_1^* is the closed unit ball in X^*. Assuming this we have:

$$\begin{aligned} \|T\| &= \sup\{\|T(x)\| : x \in B_1\} \\ &= \sup\{|f(T(x))| : x \in B_1, f \in B_1^*\} \\ &= \sup\{|T^*(f)(x)| : x \in B_1, f \in B_1^*\} \\ &= \sup\{\|T^*(f)\| : f \in B_1^*\} \\ &= \|T^*\| \end{aligned}$$

for sake of completeness we will include a proof of the fact we just used:

$$\|x\| = \sup\{|f(x)| : f \in B_1^*\}$$

note that for any $f \in B_1^*$ we have that

$$|f(x)| \leq \|f\|\|x\| \leq \|x\|$$

so we trivially have

$$\|x\| \geq \sup\{|f(x)| : f \in B_1^*\}$$

for the other inclusion. Let x be given. Define $f(x) = \|x\|$, and extend it to a linear functional using Hanh Banach, to a function such that $|f(y)| \leq \|y\|$, (meaning $f \in B_1^*$), so we have that there is a function such that $|f(x)| = \|x\|$, proving the other direction of the inequality and the result follows. $\square$

2. Theorems to know how to use

Theorem 13. ***Rouche's theorem***: *If the complex-valued functions f and g are holomorphic inside and on some closed contour K, with $|g(z)| < |f(z)|$ on K, then f and $f+g$ have the same number of zeros inside K, where each zero is counted as many times as its multiplicity*

Theorem 14. ***Krein-Milman theorem***: *Let K be a nonempty set, compact, convex, of a locally convex topological vector space X. Then K is the closed convex hull of its extreme points.*

Theorem 15. ***Krein-Milman***: *Let K be a nonempty compact, convex subset of a locally convex topological vector space X. Then K has an extreme point.*

Theorem 16. ***Alaoglu's theorem***: *X be a normed linear space. Then B^* (closed unit ball in X^*) is compact with respect to the weak-$*$ topology*

Theorem 17. ***Stone-Weirstrass***: *X locally compact Hausdorff space and A is a subalgebra of $C_0(X, \mathbb{R})$. A is dense if and only if it separates points.*

Theorem 18. ***Montel's theorem:*** *$\mathcal{F} = \{f_n\}$ be a family of holomorphic functions on Ω s.t. is uniformly bounded on compact subsets of Ω. Then, $\mathcal{F}$ is a normal family (i.e., there exists a subsequence in $\{f_n\}$ that converges uniformly on every compact set of Ω).*

Theorem 19. ***Duality of*** *L^p: Let $1 < p < \infty$ and let q be its Holder conjugate. Let (X, Ω, μ) be a measure space. For $g \in L^q$, define $F_g : L^p \to \mathbb{F}$ as follows:*

$$F_g(f) = \int fg d\mu$$

Then $F_g \in (L^p)^*$ and the map $g \mapsto F_g$ defines an isometric isomorphism of L^q onto L^p

If X is σ-finite and $g \in L^\infty$ and we define $F_g : L^1 \to \mathbb{F}$ by:

$$F_g(f) = \int fg d\mu$$

then $F_g \in (L^1)^*$ and the map $g \mapsto F_g$ defines an isometric isomorphism of L^∞ onto L^1.

Theorem 20. ***Riesz Representation Theorem:*** *If X is a locally compact space and $\mu \in M(X)$, define $F_\mu : C_0(X) \to \mathbb{F}$ by*

$$F_\mu(f) = \int f d\mu$$

Then $F_\mu \in C_0(X)^*$ and the map $\mu \to F_\mu$ is an isomorphism of $M(X)$ onto $C_0(X)^*$.

Theorem 21. ***Hanh-Banach Theorem:*** *Let X be a vector space over $\mathbb{R}$ and let q be a sublinear fucntional on X. If M is a linear manifold in X and $f : M \to \mathbb{R}$ is a linear functional such that $f(x) \leq q(x)$ for all $x \in M$, then there is a linear fnctional $F : X \to \mathbb{R}$ such that $F \mid M = f$ and $F(x) \leq q(x)$ for all $x \in X$.*

Theorem 22. ***Inverse Mapping Theorem:*** *If X and Y are Banach spaces and $A : X \to Y$ is a bounded linear transformation that is bijective, then A^{-1} is bounded.*

Remark 23. *This follows trivially from the Open Mapping theorem since if the function is bijective then we can define A^{-1} and to see that it is bounded, we just note that it is continuous since $A(U)$ is open if and only if U is open.*

Theorem 24. ***The Clsoed Graph Theorem:*** *If X and Y are Banach spaces and $A : X \to Y$ is a linear transformation such that the graph of A,*

$$graA \cong \{x \oplus Ax \in X \oplus_1 Y : x \in X\}$$

is closed, then A is continuous.

3. Problems he has asked before

1 Using calculus of residues, prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Hint: The laurent series for $\cot z = \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^3 \ldots$

Proof. Let S_N be the square with vertices at $(\pm(N+1/2), \pm(N+1/2))$ (with N a positive integer). Consider the function $f(z) = \frac{\cot \pi z}{z^4}$. It is clear that we have simple poles at the non-zero integers and that at 0 we have a pole of order 5. Applying the residue theorem we obtain:

$$\int_{S_N} f(z)dz = 2\pi i \left(\sum_{i=1}^{N} res(f,i) + res(f,-i) \right) + 2\pi i \cdot res(f,0)$$

Now we compute the residues. $res(f,0) = -\frac{\pi^3}{45}$ we can read from the Laurent expansion. To find the simple poles, let $k \neq 0$ be an integer:

$$\begin{aligned} \lim_{z \to k} (z-k) \cdot \frac{\cot \pi z}{z^4} &= \frac{1}{k^4} \cdot \lim_{z \to k} \cot \pi z(z-k) \\ &= \frac{1}{k^4} \lim_{z \to k} \frac{1}{\pi \sec^2(\pi z)} = \frac{1}{\pi k^4} \end{aligned}$$

If we manage to show that the integral on the left hand side goes to 0, then we would have:

$$0 = 2\pi i \left(\sum_{i=1}^{\infty} \frac{2}{\pi k^4} \right) + 2\pi i \left(-\frac{\pi^3}{45} \right)$$

which clearly gives the desired result. Hence, all we have to do is show:

$$\int_{S_N} f(z)dz \to 0 \qquad \text{as} \qquad N \to \infty$$

First we are going to show that in our contour S_N we have that

$$|\cot(\pi z)| \leq 2$$

for the vertical line on the right side we have that $z = (N+1/2) + iy$:

$$|\cos(\pi z)| = \frac{|\exp(i\pi((N+1/2)+iy)) - \exp(-i\pi((N+1/2)+iy))|}{2}$$

□

2 Let $A_{r_1,r_2} = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$. Show that there is a biholomorphic mapping $\varphi : A_{r_1,R_1} \to A_{r_2,R_2}$ if and only if $R_1/r_1 = R_2/r_2$. You may assume that φ extends to a homeomorphism of the closed annulus and that $\log|\varphi(z)|$ is harmonic.

Proof. One direction is completely trivial. For the other direction we will assume that $r_1 = r_2 = 1$ (by scaling), so say there is a biholomorphic map from A_1 to A_2. We will show that $R_1 = R_2$.

We first prove the following lemma. With the conditions above we have:

1 If $\lim_{|z|\to 1} |\varphi(z)| = 1$, then $\lim_{|z|\to R_1} |\varphi(z)| = R_2$

2 If $\lim_{|z|\to 1} |\varphi(z)| = R_2$, then $\lim_{|z|\to R_1} |\varphi(z)| = 1$

We shall prove this fact later. Assuming this, we can always assume that (1) happens because if (2) holds we can substitute φ for R_2/φ, and this is still a biholomorphic map from A_1 to A_2 such that the first condition is the one that holds.

Define a function $h(z) = \log|z| - \frac{\log R_1}{\log R_2} \log|\varphi(z)|$. Extend h to the closure of A_1. Since (1) is the condition that holds we have that $h(z) \to 0$ as $|z| \to 0$ and $|z| \to R_1$. Then, since we can assume that this function is harmonic, by Maximum modulus principle we have that h must be the zero function. It follows that:

$$\log|z| = \frac{\log R_1}{\log R_2} \log|\varphi(z)|$$

Hence,

$$|z|^\beta = |\varphi(z)|$$

where $\beta = \log R_2/\log R_1$. Let $P \in A_1$, and let $D_r(P)$ be a disk centered at P and with r such that $D_r(P)$ is contained in A_1. Then since we have a function that does not vanish and since the disk is simply connected we can define z^β by $e^{\log(z)\beta}$ by picking a branch of the logarithm. Hence, we have that $g(z) = \varphi(z)/z^\beta$ is a holomorphic function on the open disk $D_r(P)$, but note that $|g(z)| = 1$, so in particular the image of g is not open. By open mapping theorem, we must have that g is a constant map. That is, $\varphi(z) = e^{i\theta} z^\beta$. We can do this for each point of A_1, and since the function is continuous we have that $\varphi(z) = e^{i\theta} z^\beta$ in the entire anuli A_1. This however is only possible when β is an integer, and since φ is injective, we have that $\beta = 1$. Hence, $R_1 = R_2$. $\square$

3 Let T be the Fourier transform on $L^(\mathbb{R})$ given by

$$Tf(\zeta) = \int e^{-2\pi i x\zeta} f(x)dx \qquad \forall f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$$

What is the spectrum of T? Justify your answer.

Proof. First of all note that since $T(T(f)) = f(-x)$, we have that $T^4 = I$. Then using the identity:

$$\sigma(p(T)) = p(\sigma(T))$$

for $p(x) = x^4$, we have that $1 = (\sigma(T))^4$, so every element of $\sigma(T)$ is a fourth root of unity. To show that this are actually all the points in the spectrum, we will show that they are eigenvalues. $\{\pm 1, \pm i\}$: $\square$

4 Let μ be a finite, complex Borel measure on the real line, and suppose that for all t real,

$$\int_{-\infty}^{\infty} e^{itx} d\mu(x) = 0$$

Prove that μ is the zero measure.

Proof. $\square$

5 Let $1 < p < \infty$, and let X be a closed convex subset of $L^p([0,1], dx)$. Show that there is a point in X which is at the smallest distance from the origin.

Proof. First we will use the fact that L^p is *uniformly convex*: If for every $\epsilon > 0$ there exists a $\delta > 0$ such that for any two vectors x, y with $\|x\| = \|y\| = 1$ the condition $\|x - y\| \geq \epsilon$ implies that $\|\frac{x+y}{2}\| \leq 1 - \delta$.

Let $c = \inf\{\|x\| : x \in X\}$. If $c = 0$, then we are done, since it would imply that $0 \in X$ as X is closed. Hence, say $c > 0$, and by scaling we can assume that $c = 1$.

Let $\{x_n\}$ be a sequence such that $\{\frac{x_n}{\|x_n\|}\}$ goes to 1. We will show that this latter sequence is Cauchy. We have:

$$\frac{1}{2}\left\|\frac{x_n}{\|x_n\|} + \frac{x_m}{\|x_m\|}\right\| \geq \frac{1}{2}\|x_n + x_m\| - \frac{1}{2}\left\|\frac{x_n}{\|x_n\|} - x_n\right\| - \frac{1}{2}\left\|\frac{x_m}{\|x_m\|} - x_m\right\|$$

Note that since $\{\|x_n\|\} \to 1$ from above, we have that $\{\frac{x_n}{\|x_n\|}\} \to x_n$, so we can choose N large enough so that $n \geq N$ implies

$$\left\|\frac{x_n}{\|x_n\|} - x_n\right\| < \delta$$

also note that since X is a convex set we have that $\frac{1}{2}(x_n + x_m) \in X$, so we have $\frac{1}{2}\|x_n + x_m\| \geq 1$, thus:

$$\frac{1}{2}\left\|\frac{x_n}{\|x_n\|} + \frac{x_m}{\|x_m\|}\right\| \geq 1 - \delta$$

since we are in a uniformly convex space, we must have that

$$\left\|\frac{x_n}{\|x_n\|} - \frac{x_m}{\|x_m\|}\right\| < \epsilon$$

for $n, m \geq N$. Thus, the sequence is Cauchy. This implies that $\{x_n\}$ is Cauchy because:

$$\|x_n - x_m\| \leq \left\|x_n - \frac{x_n}{\|x_n\|}\right\| + \left\|x_m - \frac{x_m}{\|x_m\|}\right\|$$

and we can make the terms on the right arbitrary small. Thus, we have that $\{x_n\}$ converges to a point x, and since X is closed we have that $x \in X$. More over note that since norm is a continous function, we have that $1 = \lim\|x_n\| = \|\lim x_n\| = \|x\|$, so we have a point that achieves the minimum distance to the origin.

To show uniqueness of the point we will use the uniform convexity again. Say that there are two points x and x' in our set X such that they are of norm 1. Then we can find an $\epsilon > 0$ such that $\|x - x'\| > \epsilon$, but by above, we have that there exists a $\delta > 0$ such that $\|\frac{x+x'}{2}\| \leq 1 - \delta$, so in particular we have that $x + x'$ is in X, so we found a point on our set with norm less than 1, contradicting the fact that the infimum norm was 1. $\square$

6 Prove or disprove the following statement: If $\{f_n\}$ is a sequence of continous functions on $[0,1]$ which converges pointwise to a function f, then there exists a point $x_0 \in [0,1]$ such that f is continous at x_0.

Proof. For a given integer N and $\epsilon > 0$ we define the set:

$$A_N(\epsilon) = \{x : |f_n(x) - f_m(x)| \leq \epsilon \qquad \forall n, m \geq N\}$$

Note that $A_N(\epsilon)$ is a closed set. For any fixed ϵ, we have the inclusions $A_1(\epsilon) \subset A_2(\epsilon) \subset ....$ The union of this sets is all of X since we have that $f_n(x)$ converges for any fixed x, so the sequence $\{f_n(x)\}$ is Cauchy and $\mathbb{R}$ is complete. Now define:

$$U(\epsilon) = \bigcup_{N \in \mathbb{Z}_+} int(A_N(\epsilon))$$

We will first prove that $U(\epsilon)$ is open and dense in X and using the Baire category theorem we would have that the set $C = \bigcap_{n \in \mathbb{Z}_+} U(1/n)$ is dense and then we shall prove that f is continous in C.

$U(\epsilon)$ is open and dense: Let V be an arbitrary open set and we want to show that there is an N so that $V \cap int(A_N(\epsilon))$ is not empty. First of all note that the set $V \cap int(A_N(\epsilon))$ is closed in V, and since $V \subset X$, we have that V is also a Baire space, meaning that there is an m so that $V \cap int(A_m(\epsilon))$ not nowhere dense, i.e., it must contain a nonempty set W of V. Because V is open in X, the set W is open in X; therefore, it is contained in $int A_m(\epsilon)$.

f is continuous at C: Given $\epsilon > 0$, we shall find a neighborhood W of x_0 such that $|f(x) - f(x_0)| < \epsilon$ for all $x \in W$. First choose k so that $1/k < \frac{\epsilon}{3}$. Since $x_0 \in C$, we have that $x \in U(1/k)$ for a big enough k; therefore there is an N such that $x_0 \in int(A_N(1/k))$. Finally, continuity of the function f_N enables us to choose a neighborhood W of x_0, contained in $A_N(1/k)$, such that

$$|f_N(x) - f_N(x_0)| < \epsilon/3 \qquad x \in W$$

The fact that $W \subset A_N(1/k)$ implies that

$$|f_n(x) - f_N(x)| \leq 1/k \qquad n \geq N, x \in W$$

letting $n \to \infty$ we get:

$$|f(x) - f_N(x)| \leq 1/k \qquad x \in W$$

as $x_0 \in W$ we trivially have

$$|f(x_0) - f_N(x_0)| < 1/k$$

thus we have,

$$|f(x) - f(x_0)| < \epsilon \qquad x \in W$$

just as desired. $\square$

7 a Suppose $\{F_n\}$ is a sequence of functions on $L^\infty([0,1],dx)$ with norm bounded by 1, $\|F\|_\infty \leq 1$. Prove or disprove the following statement:

There is a subsequence $\{F_{n_k}\}$ such that for all $G \in L^1([0,1],dx)$,

$$\lim_{k\to\infty}\int_0^1 F_{n_k}(x)G(x)dx$$

exists.

b Suppose $\{F_n\}$ is a sequence of functions on $L^1([0,1],dx)$ with norm bounded by 1, $\|F\|_1 \leq 1$. Prove or disprove the following statement:

There is a subsequence $\{F_{n_k}\}$ such that for all $G \in L^\infty([0,1],dx)$,

$$\lim_{k\to\infty}\int_0^1 F_{n_k}(x)G(x)dx$$

exists.

Proof. a For this part we are going to use the fact that $L^\infty \cong (L^1)^*$. For each $F_n \in L^\infty$ there corresponds a linear functional in L^1, call it φ_n and moreover the functional is given by the following:

$$\varphi_n(G) = \int F_n G$$

for all $G \in L^1$. Then, by Alaoglu's theorem, we have that the closed ball of radius 1 is closed in $(L^1)^*$ is compact with respect to the weak-$*$ topology, in particular, since $\|\varphi_n\| = \|F_n\| \leq 1$, we have that $\{\varphi_n\}$ is in such ball, so there exists a convergent subsequence. $\{\varphi_{n_k}\}$, but by definition of the weak-$*$ topology, convergence of φ_{n_k} means convergence with respect to evaluation. That is, for all $G \in L^1$, $\lim_{k\to\infty}\varphi_{n_k}(G)$ converges, which is precisely what we wanted to prove.

b For a counterexample do the following construction:

□

8 Let $C \subset [0,1/2]$ be a closed subset of Lebesgue measure zero. Suppose that $f(z)$ is a bounded holomorphic function on $D\backslash C$ where $D = \{z \in \mathbb{C} : |z| < 1\}$. Prove that f can be extended to a holomorphic function on D

Proof. Define $g(z)$ as follows:

$$g(w) = \frac{1}{2\pi i}\int_{\gamma_r}\frac{f(z)}{z-w}dz$$

where γ_r is the circle of radius r and $|w| < r$ (with $1/2 < r < 1$). First of all note that $g(w)$ is well defined, that is if $r < r'$, then the definition of $g(w)$ is unambiguous as $\frac{f(z)}{z-w}$ is holomorphic in the anulus $\{z : r < |z| < r'\}$, so we can homotope γ_r to $\gamma_{r'}$ without changing the value of the integral. Hence, for any $w \in D$ we can define $g(w)$. We want to show two things: That g agrees with f in $D\backslash C$ and secondly that g is holomorphic.

For the first part: Let $w \in D \backslash C$. Then since C is closed there exists an open ball B with $w \in B$ and $B \subset D \backslash C$. $\square$

9 Find with proof the number of zeros of the function $9z^6 + e^{2z}$ on the closed unit disc in the complex plane.

Proof. Let $f = 9z^6$ and let $g = e^{2z}$, then for $|z| = 1$, we have that $|f(z)| = 9$ and $|g(z)| = |e^{2z}| = |e^{2Re(z)}| \le e^2 < 9$, then we have that f (which clearly has 6 zeroes, has the same number of zeroes as $9z^6 + e^{2z}$ by Rouche's theorem. $\square$

10 Let X be a Banach space, and let A and B be closed linear subspaces.

a Assume that

$$\inf\{\|x - y\| : x \in A, y \in B, \|x\| = \|y\| = 1\} = \delta > 0$$

Show that $A + B$ is closed in V.

b Assume that $A + B = V$ and $A \cap B = \{0\}$. Show that the condition in part a must be true.

Proof. The idea of the proof will be the bound

$$c(\|a\| + \|b\|) \le \|a + b\|$$

because if we do so, then we would have the following: Let $\{c_n\}$ be Cauchy in $C = A + B$, so rewrite it as $\{c_n\} = \{a_n + b_n\}$ with $a_n \in A$ and $b_n \in B$. Then we would have that for large n, m:

$$\epsilon > \|a_n + b_n - (a_m + b_m)\|$$

$$= \|a_n - a_m + b_n - b_m\| \ge c_1(\|a_n - a_m\| + \|b_n - b_m\|) \ge c_1\|a_n - a_m\|$$

hence $\{a_n\}$ is Cauchy and so is $\{b_n\}$. As A and B are closed, we have that $a_n \to a$ and $b_n \to b$ with $a \in A$ and $b \in B$, so we have that $a_n + b_n \to a + b \in A + B$. That is, $A + B$ is closed.

To show that $c(\|a\| + \|b\|) \le \|a + b\|$ we will proceed as follows: First we assume that $\|a\| + \|b\| = 1$ and we show that:

$$\inf\{\|a + b\| : \|a\| + \|b\| = 1\} \ge c := \frac{\min\{\delta, 1\}}{4} \tag{1}$$

First of all note that if $|\ \|a\| - \|b\|| \ge c$, then we have that $\|a + b\| \ge \|a\| - \|b\|$ and $\|a + b\| \ge \|b\| - \|a\|$, so we have that $\|a + b\| \ge |\|a\| - \|b\|| \ge c$. Thus, assume that $|\|a\| - \|b\|| < c$. Note that this implies that $a \ne 0$ because otherwise we would have that $0 + \|b\| = \|a\| + \|b\| = 1$ and $\|b\| < c$ so we would have $1 < c$ a contradiction with the definition of c (this is why in the definition of c we use the $\min\{1, \delta\}$, that way we ensure that both a and b are non-zero. Hence,

$$\begin{aligned}
\|a+b\| &= \left\| a - \frac{a}{2\|a\|} + \frac{a}{2\|a\|} + b - \frac{b}{2\|b\|} + \frac{b}{2\|b\|} \right\| \\
&\geq \left\| a - \frac{a}{2\|a\|} \right\| + \left\| \frac{a}{2\|a\|} - \frac{b}{2\|b\|} \right\| + \left\| b - \frac{b}{2\|b\|} \right\| \\
&= -\big| \|a\| - \frac{1}{2} \big| + \| \frac{a}{2\|a\|} - \frac{b}{2\|b\|} \| - \big| \|b\| - \frac{1}{2} \| \\
&\geq \frac{\delta}{2} - \big| \|a\| - \frac{1}{2} \big| - \big| \|b\| - \frac{1}{2} \| \\
&= \frac{\delta}{2} - |\|a\| - \|b\|| > \frac{\delta}{2} - c > \frac{\delta}{2} - \frac{\delta}{4} = \frac{\delta}{4} > c
\end{aligned}$$

so we see that (1) is proven. To see how this implies that $c(\|a\| + \|b\|) \leq \|a+b\|$ for general a and b, just note:

$$\left\| \frac{a}{\|a\| + \|b\|} + \frac{b}{\|a\| + \|b\|} \right\| \geq c$$

multiplying both sides by $(\|a\| + \|b\|)$ gives the desired result.

For the second part first let Q be the quotient map by A. That is $Q : X \to X/A$. Then since A is closed we have that X/A is a Banach space, and moreover, we have that the restriction of Q to B, call it T will be surjective (as $A+B = X$) and injective (as $A \cap B = \{0\}$). Then by the Banach open theorem, we have that T is an open map. This is equivalent of saying that T^{-1} is continuous. Now assume for sake of contradiction that

$$\inf\{\|a-b\| : a \in A, b \in B, \|x\| = \|y\| = 1\} = 0$$

then we can find a sequence $\{a_n - b_n\}$ such that $\|a_n\| = 1 = \|b_n\|$ with $a_n \in A$ and $b_n \in B$ and $a_n - b_n \to 0$. Then note the following: $a_n - b_n \to 0$ implies that $Q(a_n - b_n) \to 0$. Hence, $\overline{a_n - b_n} \to \overline{0}$. Thus we have:

$$0 = T^{-1}(\overline{0}) = T^{-1}(\lim \overline{a_n - b_n}) = \lim(T^{-1}(\overline{a_n - b_n})) = \lim(T^{-1}(\overline{b_n})) = \lim(b_n)$$

a contradiction since $\|b_n\| = 1$, so their limit cannot be 0. $\square$

11 Let ℓ^2 be the Hilbert space of sequences $\alpha = \{a_n\}, n \leq 1$, such that $\sum |a_n|^2$ converges, with the hermitian product

$$\langle \alpha, \beta \rangle = \sum a_n \overline{b_n}$$

Let T be the shift operator, that is

$$T(\alpha) = (0, a_1, a_2, ..)$$

Compute the spectrum of T.

12 For functions $\phi \in C_0^\infty(\mathbb{R})$ define the principal value integral against $1/x$ by

$$\lim_{\epsilon \to 0} \int_{\epsilon < |x| < 1/\epsilon} \phi(x) \frac{1}{x} dx = \langle \phi(x), \frac{1}{x} \rangle$$

Show that $\langle \phi(x), \frac{1}{x} \rangle$ extends to a tempered distribution and compute the Fourier transform of this extension.

Proof. We need to show that for f in the Schwarz space we have that

$$\lim_{\epsilon \to 0} \int_{\epsilon < |x| < 1/\epsilon} f(x) \frac{1}{x} dx$$

exists. Write the integral as follows:

$$\int_{\epsilon < |x| < 1} \frac{f(x) - f(0)}{x - 0} dx + \int_{\epsilon < |x| < 1} \frac{f(0)}{x - 0} dx + \int_{1 \leq |x| < 1/\epsilon} \frac{f(x)}{x} dx$$

First of all note that the middle term is zero since the function $1/x$ is odd. Secondly, the third term converges as f is rapidly decreasing (in particular, we have that $f(x)/x \leq 1/x^2$ for large values of x, so we have convergence for the integral comparison test). Lastly, by the intermediate value theorem for derivatives, there is a point $c_x \in (0, x)$ such that $\frac{f(x)-f(0)}{x-0} = f'(c_x)$, but note that f' is going to be continuous, and in the interval $[0, 1]$ we have that $|f'| \leq M$, so we have that the first integral is finite as well.

To find the Fourier transform: $\square$

13 Using calculus of resides, prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = -\frac{7\pi^4}{720}$$

Hint: Consider the function $f(z) = \frac{1}{\sin \pi z}$

11 Let H be the space of all analysis functions F on the open unit disk with norm

$$\|F\|_H = \left(\int \int |F(z)|^2 dx dy \right)^{1/2} < \infty$$

a Prove that H is complete in this norm.

b Exhibit with proof a complete orthonormal basis for H

Proof. a Let $\{f_n\}$ be a Cauchy sequence in H. First of all note that $f_n \in L^2$, which is a Banach space. Hence, we have that $f_n \to f$ in the L^2 norm to a function f. We shall show that f is indeed holomorphic.If we manage to show that $\{f_n\}$ is bounded in compact subsets of D, then we will be able to apply Montel's theorem, which will yield a subsequence of functions that converge uniformly in every compact subset of D. Since uniform limit of holomorphic functions is holomorphic, we would have that f is indeed holomorphic (the subsequence must have to converge to f).

Hence, we show that $\{f_n\}$ is bounded in compact subsets of D: Let K be a compact set of D, and let D_r be a closed disk containing K. We shall show that f is bounded in D_r instead. First note that since D has finite measure we have that $\|f\|_1 \leq A\|f\|_2$ for some constant A and all $f \in L^2$. Also, since $\{f_n\}$ converge in the L^2 norm, we have that there exists an M such that $\|f_n\|_2 \leq M$

for all n. Hence, $\|f_n\|_1 \leq AM$ for all n. We claim that $\sup_{z\in D_r}|f(z)| \leq B\|f\|_1$ for some B (putting all this together would give the desired bound).

$$\|f\|_1 = \int_D |f(z)|dz = \int_0^{2\pi}\int_0^1 |f(re^{i\theta})|rdrd\theta$$

let r' be such that $r < r' < 1$. Then we have that:

$$f(z) = \frac{1}{2\pi i}\int_{C_{r'}} \frac{f(w)}{w-z}dw \Rightarrow |f(z)| \leq \frac{1}{2\pi}\int_{C_{r'}} \frac{|f(w)|}{|w-z|}dw$$

as $w \in C_{r'}$, so we have that $|w - z| > \delta$ for some $\delta > 0$. Thus,

$$|f(z)| \leq \frac{1}{2\pi\delta}\int_{C_{r'}} |f(z)|dz = \frac{1}{2\pi\delta}\int_0^{2\pi} |f(r'e^{i\theta})|r'd\theta$$

this holds for any radius greater than r', so integrate both sides when the radius goes from r' to 1:

$$(1-r')|f(z)| \leq \frac{1}{2\pi\delta}\int_{r'}^1 |f(re^{i\theta})|rd\theta dr$$

Thus,

$$|f(z)| \leq \frac{1}{2\pi\delta(1-r')}\int_{r'}^1 |f(re^{i\theta})|rd\theta dr \leq \frac{1}{2\pi\delta(1-r')}\int_0^1 |f(re^{i\theta})|rd\theta dr = \frac{\|f\|_1}{2\pi\delta(1-r')}$$

Hence, for f holomorphic we have that $\sup_{z\in K}|f(z)| \leq B\|f\|_1$ for a constant B. Hence we have the desired bound, and we are done by earlier remarks.

b

□

14 Find with proof the number of zeroes of the function $8z^4 + e^{2z}$ on the closed unit disc in the complex plane.

Proof. Let $f = 8z^4$ and $g = e^{2z}$, then we have that $|g(z)| < |f(z)|$ for all z in the unit circle, so $f + g$ has the same number of zeroes than f, which clearly has 4 (By Rouche's theorem). □

15 Let f be differentiable at every point in $[0,1]$. Prove that there exists $x_0 \in [0,1]$ such that $f'(x)$ is continuous at x_0.

Proof. Define $f_n(x) = \frac{f(x)-f(x+1/n)}{1/n}$, then we have that $f_n(x) \to f_n'(x)$ pointwise. Note that each f_n is continuous, so by problem 6, we have that there exists a point x_0, such that $f'(x_0)$ is continuous. □

4. Topics on Functional analysis

4.1. **Spectrum of an operator.** The spectrum of an operator is the generalization of eigenvalues that we learned in linear algebra. Recall, in linear algebra, we said that λ was an eigenvalue for T if the operator $T - \lambda I$ was not injective. In reality, we are concerned with the set of λ such that $T - \lambda I$ is not invertible. Since in finite dimensions we have $\dim V = \dim\ker(T) + \dim ran(T)$, we have that $T - \lambda I$ was not surjective either. Note that this is not the case in infinite dimensions. We can have a λ such that $T - \lambda I$ is injective, but fails to be surjective (consider the shift operator to the right). Hence, we have to develop some notions of different kinds of spectrum.

Definition 25. *Let T be a bounded operator on a Banach space X. The spectrum of T, $\sigma(T)$, consists of all the scalars λ such that $\lambda I - T$ does not have an inverse that is a bounded operator. In this case, this is equivalent to the set of λ such that $\lambda I - T$ is not bijective.*

It is convenient to note that the spectrum of an operator can be decomposed into the following three sets:

1 $\sigma_p(T)$: **Point spectrum:** The set of λ such that $\lambda I - T$ is not injective. Meaning, this is the set of eigenvalues.
2 $\sigma_r(T)$: **Residual spectrum:** The set of λ such that $\lambda I - T$ is injective, but does not have a dense range.
3 $\sigma_c(T)$: **Continuous spectrum** The set of λ such that $\lambda I - T$ is injective and has a dense range, but the range fails to be closed (meaning that it is not surjective)

Note that by definition we then have the following:

$$\sigma(T) = \sigma_p(T) \cup \sigma_r(T) \cup \sigma_c(T)$$

where the unions above are disjoint.

Lemma 26. *$\sigma(T)$ is always a closed, bounded, non-empty set of the complex numbers.*

5. Topics on Complex analysis

Theorem 27. ***The open mapping theorem:*** *If $f : U \to \mathbb{C}$ is a nonconstant holomorphic function on a connected open set U, then $f(U)$ is an open in $\mathbb{C}$*

Theorem 28. ***Maximum modulus principle:*** *Let $U \subset \mathbb{C}$ be a domain. Let f be a holomorphic function on U. If there is a point $P \in U$ such that $|f(P)| \geq |f(z)|$ for all $z \in U$, then f is constant.*

Proof. Assume that there is such a P. If f is not constant, by above we have that f is an open map. In particular there should be a neighborhood around $f(P)$ such that it is contained in $f(U)$, but this implies that there is a $\eta \in f(U)$ with $|\eta| > |f(P)|$ a contradiction. □

Theorem 29. ***Maximum modulus principle:*** *Let $U \subset \mathbb{C}$ be a bounded domain. Let f be a continuous function on $\overline{U}$ that is holomorphic on U. Then the maximum value of $|f|$ on $\overline{U}$ must occur in δU.*

Proof. As $|f|$ is continuous on the compact set $\overline{U}$ then it achieves its maximum. If $|f|$ is constant there is nothing to prove, so assume $|f|$ is not constant, then the maximum value of $|f|$ cannot happen in U, by our last theorem, so it must happen in δU. □

The above has two important consequence:

(1) If $U \subset \mathbb{C}$ is a domain, f holomorphic in U, if there exists a P such that $|f|$ has a local maximum at P, then f is constant.
(2) If $U \subset \mathbb{C}$ is a domain, with $f(z) \neq 0$ for all $z \in U$. If there exists a $P \in U$ such that $|f(P)| \leq |f(z)|$ for all $z \in U$, then f is constant.

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